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LETTER TO THE EDITOR

The theory of coupled differential equations in supersymmetric quantum mechanics

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Abstract. There is a close relationship between the theory of coupled differential equations and supersymmetric quantum mechanics. We set up the bridge connecting these fields with two theorems concerning both the coupling and non-coupling cases.

Since the introduction of the concept of shape-invariant potentials (SIP) and its connection with the factorization approach of the Schrödinger equation (Gedenshtein 1983, Infeld and Hull 1951) there has been widespread interest in various aspects of SUSY. During the last two years, a number of papers have focused on the bound states problem in dealing either with the quest for families of exactly solvable potentials and their energy spectrum or the determination of the excited states wavefunction by repeated use of the 'ladder operator' technique (see, for example, Cooper *et al* 1989, Stahlhofer and Bleuler 1989, Dutt *et al* 1988, Keung *et al* 1989, Montemayor and Salem 1989, Fernandez *et al* 1989 and a more recent review by Lahiri *et al* 1990).

There is, on the other hand, another aspect which also deserves some more attention, namely the SUSY formulation in the 'continuum' ($E>0$ or scattering states) of the Schrödinger equation (see, for instance, Sukumar 1987, Amado *et al* 1988, Cooper *et al* 1988, Khare and Sukhatme 1988).

In this letter we shall mainly discuss this second aspect by noting that the scattering problem within the two-state approximation is described by a system of two coupled differential equations which bear a strikingly close similarity to the mathematical formulation of SUSY in the $N=I$ case. To simplify, we continue to keep the previous notations (Cao 1981) in writing the system of CDE as:

$$[P+f_1(x)]\phi_1=B(x)\phi_I \qquad [P+f_2(x)]\phi_2=B(x)\phi_1 \tag{1}$$

in which $P=\mathrm{d}^2/\mathrm{d}x^2, f_i(x)(i=I,2)$, $B(x)$ can be any analytic functions of x. On the other hand, in supersymmetry we are concerned with a system of two coupled differential equations of first order:

$$A^+\phi_1=2^{1/2}\bar{k}\phi_2 \qquad A^-\phi_2=2^{1/2}\bar{k}\phi_1 \tag{2}$$

with $\bar{k}^2=k_0^2-a$, $\bar{k}^2$ is the energy (eigenvalue), a is a constant which may depend on a parameter, ϕ_1, ϕ_2 are the bosonic and fermionic components, $A^\pm$ are the 'ladder' operators defined by:

$$A^\pm=\pm\frac{\mathrm{d}}{\mathrm{d}x}+v'(x) \tag{3}$$

$v'(x)=\mathrm{d}v(x)/\mathrm{d}x$, $v(x)$ is the superpotential, k_0^2 incident energy.

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The Hamiltonian is now a 2×2 diagonal matrix $H(H_+, H_-)$ with

$$2H_+ = A^-A^+ = -\frac{d^2}{dx^2} + v'^2 - v'' \qquad 2H_- = A^+A^- = -\frac{d^2}{dx^2} + v'^2 + v'' \tag{4}$$

$$[H_\pm - \bar{k}^2]\phi_{1,2} = 0. \tag{5}$$

For the case where $B(x) = 0$ (no coupling), the relationship mentioned above can be formulated by the following theorem.

Theorem 1. Let the analytical structure of $f_i(x)$, $i = 1, 2$ be such that

$$f_i(x) = k^2 - h_i(x) \tag{6}$$

$h_i(x)$ satisfying the following condition ($k^2 = 2\bar{k}^2$):

$$[h_1(x) + h_2(x)]^{1/2} = 2^{-1/2} \int^x (h_1 - h_2)\, dx \tag{7}$$

then,

(a) the solutions ϕ_1, ϕ_2 of the Schrödinger equations (1) are partners in the sense of supersymmetry;

(b) their differentiation ϕ_1', ϕ_2' can be expressed by a 'recurrence relation' in terms of ϕ_1, ϕ_2.

Proof. (a) When $B(x) = 0$, we may always write (1) as:

$$\left[\frac{d^2}{dx^2} + k^2 - \frac{1}{2}(h_1 + h_2) \mp \frac{1}{2}\Delta f\right]\phi_{1/2} = 0 \tag{8}$$

where $\Delta f = f_1 - f_2$. From (4), (5), (7) we have

$$\tfrac{1}{2}(h_1 + h_2) = v'^2 \qquad \tfrac{1}{2}(f_1 - f_2) = v''. \tag{9}$$

(b) As ϕ_1, ϕ_2 are partners, from (2) we see that:

$$\phi'_{1,2} = \mp \frac{1}{2^{1/2}}(h_1 + h_2)^{1/2}\phi_{1,2} \pm k\phi_{2,1}. \tag{10}$$

As an example, we shall consider the cases of one and two parameters.

One parameter. Let

$$h_i(x) = r_i - s_i[f(x)]^n \tag{11}$$

r_i, s_i are constants and may depend on a parameter, $f(x)$ is the unknown function. We consider for instance the following cases:

1. $k \neq 0$, $n = 1$, $r_i = 0$, $s_1 = l(l+1)$, $s_2 = (l+1)(l+2)$;
2. $k = 0$, $n = 2$, $r_1 = \frac{1}{4}$, $r_2 = -\frac{1}{4}$, $s_1 = s_2 = \frac{1}{2}$;
3. $k = 0$, $n = 1$, $r_1 = r_2 = \alpha^2$, $s_1 = -\alpha(\alpha - 1)$, $s_2 = -\alpha(\alpha + 1)$.

It can be verified that using (7), the function $f(x)$ must be solution of the following corresponding differential equations:

$$f'f^{-3/2} = 1 \qquad f' = 1 \qquad \frac{f'}{f\sqrt{1-f}} = -2$$

with solutions of the form:

$$f \simeq x^2 \qquad f \simeq x \qquad f \simeq \operatorname{sech}^2 x.$$

In case (1) the components (ϕ_1, ϕ_2) are simply the Bessel functions $\sqrt{x}J_{l+1}(kx)$, $\sqrt{x}J_{l+3/2}(kx)$ with well known recurrence relations as can also be checked using (10). Case (2) is the usual oscillator problem while case (3) corresponds to the Pösch–Teller potential $(v'(x) = \alpha \tanh x)$ with eigenvalue $-\alpha^2/2$.

Two parameters. Let

$$v'(x) = c\frac{\sqrt{1+f^2}}{f} - \frac{b}{f}$$

with $f' = \sqrt{1+f^2}$ so that $f = \sinh x$. From (7) we have (c and b are parameters):

$$h_{1,2}(x) = \frac{c^2}{2} + \frac{1}{2\sinh^2 x}[c(c\pm 1) + b^2 - b(2c\pm 1)\cosh x] \tag{12}$$

with eigenvalue $k^2 - c^2/2$.

The condition (7) which ensures the existence of a superpotential is valid in all cases. If, on the other hand, shape invariance is assumed, the functions $f_i(x)$ must also be related by another additional relation first pointed out by Gedenshtein (1983)

$$f_1(p_0, x) = f_2(p_1, x) + C(p_1)$$

(p_0 = parameter, $p_1 = g(p_0)$).

For $k = 0$, we have indeed verified that (7) does incorporate all known exactly solvable shape-invariant potentials (Dutt *et al* 1988). The question of whether it may also include some special type of exactly solvable, but non-shape, invariant potentials is under investigation.

For further application, it will be interesting to add a 'Coulomb' term in case (1) so that

$$h_1(x) = -\frac{1}{x} + \frac{l(l+1)}{x^2} \qquad h_2(x) = -\frac{1}{x} + \frac{(l+1)(l+2)}{x^2}$$

then

$$v'(x) = \frac{l+1}{x} - \frac{1}{2(l+1)}.$$

If we now include explicitly the charge Z in this formulation, the resulting couple (H_+, H_-) defined by Roy (1990)

$$H_+ = -\frac{d^2}{dx^2} - \frac{2Z}{x} - \frac{l(l+1)}{x^2} + \frac{Z^2}{(l+1)^2} \qquad H_- = -\frac{d^2}{dx^2} - \frac{2Z}{x} + \frac{(l+1)(l+2)}{x^2} + \frac{Z^2}{(l+1)^2} \tag{13}$$

are SUSY partners according to (7). However, the alternative couple ($\bar{H}_+$, $\bar{H}_-$ derived from the 'ladder operator' technique (N: principal quantum number

$$\bar{H}_+ = -\frac{d^2}{dx^2} - \frac{2Z}{x} + \frac{l(l+1)}{x^2} + \frac{Z^2}{N^2} \qquad \bar{H}_- = -\frac{d^2}{dx^2} - \frac{2Z}{x}\left(1 - \frac{1}{H}\right) + \frac{l(l+1)}{x^2} + \frac{Z^2}{N^2} \tag{14}$$

cannot, strictly speaking, be regarded as SUSY partners because of the non-validity of condition (7) (or equivalently non-existence of the appropriate superpotential).

The coupling case (4). When $B(x) \neq 0$, the original representation (ϕ_1, ϕ_2) become inadequate and we shall need a new representation $(\bar{\phi}_1, \bar{\phi}_2)$ defined by

$$\bar{\phi} = T(a)\phi \qquad T(a) = \begin{pmatrix} 1-a & 1+a \\ -(1+a) & 1-a \end{pmatrix}. \tag{15}$$

From the theorem of separation (Cao 1981), we also know that in the representation $\bar{\phi}$, the equations are completely decoupled if f_1, f_2 and $B(x)$ are such that $B(x)/\Delta f = \alpha$ where $\alpha =$ constant and

$$a = -2\alpha + m \qquad m = \sqrt{1+4\alpha^2}. \tag{16}$$

Its connection with supersymmetry can be formulated by a second theorem.

Theorem 2. Consider system (1) and assume that the conditions (7) and (16) are simultaneously valid. Then:

(a) the components $\bar{\phi}_1$, $\bar{\phi}_2$ are not partners;

(b) under certain conditions, it is possible to generate two families of superpotentials such that $\bar{\phi}_1$, $\bar{\phi}_2$ have their own partners $\bar{\bar{\phi}}$, $\bar{\bar{\phi}}_2$.

Proof. (a) The separated equations can be written as:

$$\left[\frac{d^2}{dx^2} + \frac{1}{2}(f_1+f_2) \pm \frac{1}{2}(f_1-f_2)\right]\bar{\phi}_{1,2} = \mp\frac{1}{2}(m-1)(f_1-f_2)\bar{\phi}_{1,2}. \tag{17}$$

Consider now the LHS of (17) and let $\bar{\phi}_1^{(0)}$, $\bar{\phi}_2^{(0)}$ be solutions of this homogeneous differential equation. From theorem 1 we already know that $\bar{\phi}_1^{(0)}$, $\bar{\phi}_2^{(0)}$ are partners. The existence of the RHS term ($m > I$) will invalidate condition (7) so the two components $\bar{\phi}_1$, $\bar{\phi}_2$ cannot generally be supersymmetric partners except however for some very special cases. Note also that the inclusion of a coupling can also be regarded as equivalent to a rotation of an angle φ ($\tan\varphi = (1-a)/(1+a)$) in the abstract space $(\bar{\phi}_1, \bar{\phi}_2)$ (Cao 1988).

(b) Let $A = \int^x (f_1 - f_2)\, dx$ and let a matrix $\bar{v}(x) = (\bar{v}_1, \bar{v}_2)$ be defined by

$$\bar{v}'_{1,2} = \pm(\tfrac{1}{2}A + \psi) \tag{18}$$

$\psi(x)$ being for the moment arbitrary.

From (17) we may now subject $\psi(x)$ to the condition:

$$\tfrac{1}{4}A^2 \pm \tfrac{1}{2}mA' = (\tfrac{1}{2}A + \psi)^2 \pm (\tfrac{1}{2}A' + \psi') \tag{19}$$

so that it can be determined by the Ricatti equation:

$$\pm\psi' + A\psi + \psi^2 \mp \tfrac{1}{2}(m-1)A' = 0 \tag{20}$$

with two solutions corresponding to the $\pm$ sign.

In order to solve (20) we must have recourse to standard methods which consist of two successive transformations. First we set $\psi_i = \varepsilon_i (d/dx) \log \mu_i$ and secondly $\mu_i = c_i(x) \exp[\frac{1}{2}\varepsilon_i \int^x A\, dx]$, $\varepsilon_i = \pm 1$, $i = 1, 2$.

It can be verified that $c_i(x)$ are solutions of the following equations:

$$\left[\frac{d^2}{dx^2} - \frac{1}{4}(A^2 + 2mA')\right]C_1 = 0 \qquad \left[\frac{d^2}{dx^2} - \frac{1}{4}(A^2 - 2(m-2)A')\right]C_2 = 0. \tag{21}$$

From (18) we see that the two components of the superpotential are simply $\bar{v}'_1 = c'_1/c_1$, $\bar{v}'_2 = c'_2/c_2$ and the conditions mentioned above concern the existence of solutions of these equations, solutions which are determined up to a constant of integration. □

We note that these equations become identical if $m=1$ (no coupling). Asymptotically the quantity $A(x)$ is generally a constant. For instance in the Pösch-Teller case $A(x)=2\alpha \tanh x$ so that $A(\infty)=2\alpha$. The asymptotic solutions are $c_i(\infty)=\exp(\pm A)$. If we assign the + (−) signs to c_1 (c_2) then the superpotential $\bar{v}'$ is asymptotically reduced to a constant diagonal matrix

$$\bar{v}'(\infty)=\begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}.$$

This is precisely the result obtained by Amado *et al* (1988) using another approach which is valid only for the case of threshold difference (non-resonance) while in the present one the resonance case is taken into account.

The appropriate coupling functions $B(x)$ can be determined from the condition $B(x)/\Delta f(x)=cte$. For instance in the four cases (1)-(4) above the coupling functions are respectively ($B_0=cte$): $B(x)=B_0x^{-2}$, $B(x)=cte$, $B(x)=B_0\,(\cosh^2 x)^{-1}$, $B(x)=B_0\,(\sinh^2 x)^{-1}c-b\cosh x$. The non-resonance case can also be dealt with by use of the technique of auxiliary parameters (Cao 1982, 1988).

The following example can be regarded as a pedagogical hint for the use of the last theorem because of its relative simplicity. Consider in fact the case of Bessel functions discussed above (one parameter, case 1). We have for this case $A=\gamma x^{-1}$, $\gamma=2(l+1)$ so that it would be more appropriate to solve (19) directly with a solution of the form $\psi=a_i x^{-1}$, $i=1, 2$ where

$$a_1=(l+\tfrac{3}{2})\left[1-\left(1+\frac{4(m-1)(l+1)}{(2l+3)^2}\right)^{1/2}\right]$$
$$a_2=(l+\tfrac{1}{2})\left[1-\left(1-\frac{4(m-1)(l+1)}{(2l+1)^2}\right)^{1/2}\right]. \qquad (22)$$

It is possible in this special case to ascribe to the quantities a_i a precise physical meaning. Indeed, if there is no coupling ($m=1$), $a=0$ and the asymptotic forms of $\bar{\phi}_1^{(0)}$, $\bar{\phi}_2^{(0)}$ are $\sin(kx-l\pi/2)$ and $\sin(kx-(l+1)\pi/2)$. In the presence of a coupling ($m>1$) they are $\sin(kx-(l+a_1)\pi/2)\simeq\bar{\phi}_1$ and $\sin(kx-(l+1+a_2)\pi/2)\simeq\bar{\phi}_2$. Therefore the quantities $\bar{\delta}_{11}=a_1\pi/2$ and $\bar{\delta}_{22}=a_2\pi/2$ can be considered as the eigenphase shift in the ($\bar{\phi}$) representation (Cao 1984). From (10) we also see that their partners are respectively $\bar{\bar{\delta}}_{11}=(a_1+1)\pi/2$ and $\bar{\bar{\delta}}_{2,2}=(a_2+1)\pi/2$.

As the system of differential equations is decoupled in the ($\bar{\phi}$) representation, the corresponding $\bar{S}$ matrix ($\bar{S}=\exp(\mathrm{i}\bar{\delta})$ is a diagonal matrix

$$\bar{S}=\begin{pmatrix}\bar{S}_{11} & 0\\ 0 & \bar{S}_{22}\end{pmatrix}.$$

The original S matrix in the (ϕ) representation is simply:

$$S=T_{(a)}^{-1}\bar{S}T_{(a)}.$$

Physically, this example can be associated with the 'optical resonance' transition $|n,l\rangle\rightarrow|n,l\pm1\rangle$ in $e-H$ scattering with the Lane-Lin model (1964) in which, at first approximation, the Coulomb effect is neglected and only the effect of a dipole coupling is taken into account in the frame of a two-state approximation.

It is also possible to generalize the Lane-Lin model by incorporating the Coulomb term in (17). We briefly present here some results which can be easily verified. For instance the two components of the matrix $\bar{S}$ are now:

$$\bar{S}_{11}=\frac{\Gamma[l+1+a_1-(2ik_1)^{-1}]}{\Gamma[l+1+a_1+(2ik_1)^{-2}]} \qquad S_{2,2}=\frac{\Gamma(l+2+a_2-(2ik_2)^{-1}]}{\Gamma(l+2+a_2+(2ik_2)^{-1}|} \tag{23}$$

in which $k_1^2=k^2-[4(l+1+a_1)^2]^{-1}$; $k_2^2=k^2-[4(l+1+a_2)^2]^{-1}$. They are related to their corresponding partners by:

$$\bar{S}_{1,1}=\frac{ik_2-W_1}{ik_1+W_1}\bar{\bar{S}}'_{1,1} \qquad \bar{\bar{S}}_{2,2}=\frac{ik_2-W_2}{ik_2+W_l}\bar{S}_{2,2} \tag{24}$$

where

$$W_1=\frac{1}{2(l+1+a_1)} \qquad W_2=\frac{1}{2(l+1+a_2)}.$$

To summarize, we may conclude that the above discussion clearly suggests a close formal relationship between the basic ideas of supersymmetric quantum mechanics on one hand and the theory of coupled differential equations on the other. Condition (7) in this sense, does play an essential role in this connection because it ensures the existence of the appropriate superpotential. The possibility of including a coupling in the theory may also open new perspectives in widening the range of future investigations.

References

Amado R D, Cannata F and Delonder J P 1988 *Phys. Rev.* A 7 3797
Cao X C 1981 *J. Phys. A: Math. Gen.* **14** 1069
—— 1982 *J. Phys. A: Math. Gen.* **15** 2727
—— 1984 *J. Phys. A: Math. Gen.* **17** 609
—— 1988 *J. Phys. A: Math. Gen.* **21** 617
—— 1990 *J. Phys. A: Math. Gen.* **23** L659
Cooper F, Ginocchio J M and Khare A 1989 *J. Phys. A: Math. Gen.* **22** 3707
Cooper F, Ginocchio J M and Wipf A 1988 *Phys. Lett.* **129A** 145
Dutt R, Khare A and Sukhatme U P 1988 *Am. J. Phys.* **56** 163
Fernandez F M, Ma Q, Desmet D J and Tipping H H 1989 *Can. J. Phys.* **67** 931
Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
Keung W T, Sukhatme U P, Wang Q and Imbo T D *J. Phys. A: Math. Gen.* **22** L983
Gedenshtein L E 1983 *JETP Lett.* **38** 356
Khare A and Sukhatme U P 1986 *J. Phys. A: Math. Gen.* **21** L501
Lahiri A, Roy P K and Bagchi B 1990 *Int. J. Mod. Phys.* A **5** 8
Lane N F and Lin C C 1964 *Phys. Rev.* A **133** 947
Levai G 1989 *J. Phys. A: Math. Gen.* **22** 689
Montemayor R and Salem L D 1989 *Phys. Rev.* A **40** 2170
Roy P K 1990 *Mod. Phys. Lett.* A **5** 67
Stalhofer A and Bleuler K 1989 *Nuovo Cimento* B **104** 447
Sukumar C V 1987 *J. Phys. A: Math. Gen.* **20** 2461